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Universiti Tun Hussein Onn Malaysia

UNIVERSITI TUN HUSSEIN ONN MALAYSIA

FINAL EXAMINATION
SEMESTER II
SESSION 2023/2024

COURSE NAME	:	DIFFERENTIAL EQUATIONS
COURSE CODE	:	BDA 24303
PROGRAMME CODE	:	BDD
EXAMINATION DATE	:	JULY 2024
DURATION	:	3 HOURS
INSTRUCTIONS	:	1. PART A: ANSWER **THREE (3)** QUESTIONS FROM FOUR QUESTIONS ONLY PART B: ANSWER ALL QUESTIONS 2. THIS FINAL EXAMINATION IS CONDUCTED VIA ☐ Open book ☒ Closed book 3. STUDENTS ARE **PROHIBITED** TO CONSULT THEIR OWN MATERIAL OR ANY EXTERNAL RESOURCES DURING THE EXAMINATION CONDUCTED VIA CLOSED BOOK

THIS QUESTION PAPER CONSISTS OF **EIGHT (8)** PAGES

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PART A: ANSWER ANY THREE (3) QUESTIONS

Q1 Solve the following first order ordinary differential equations with the initial value $y(0) = 1$:

(a) $\frac{dy}{dx} = \frac{y^2+x^2}{2xy}$

(10 marks)

(b) $\frac{dy}{dx} = -\frac{2xy}{4y+x^2}$

(10 marks)

Q2 Given the second order non-homogeneous ordinary differential equation:

$$y'' - 2y' + y = 5e^x$$

Taking $y(0) = 1$ and $y'(0) = 2$, perform the following:

(a) Solve the initial value problem by using the method of undetermined coefficients.
(10 marks)

(b) Solve the same problem by using the method of variation of parameters.
(10 marks)

Q3 (a) Solve the following integral:

$$\int_0^\infty \frac{e^{-3t}\sin^2(t)}{t}dt$$

*Hint: The integration can be performed using Laplace transform.

(6 marks)

(b) Sketch the graph and express the following function in terms of unit step functions and find the corresponding Laplace transform.

$$f(t) = \begin{cases} t, & 0 \le t \le 3 \\ 3 - t, & 3 \le t < 6 \\ 0, & t \ge 6 \end{cases}$$

(14 marks)

Q4 A second order non-homogeneous ordinary differential equation is given as:

$$y'' - 5y' + 6y = 10\cos(t)$$

with initial conditions $y(0) = 1$ and $y'(0) = \beta$

(a) Find the Laplace transform and arrange the resulting equation in terms of $Y(s)$.

(5 marks)

(b) Continue to solve the equation using partial fractions and the inverse Laplace transform.

(10 marks)

(c) Taking the initial conditions for the problem **Q4** to be $y(0) = 1$ and $y'(0) = 1$, calculate the particular solution.

(5 marks)

PART B: ANSWER ALL QUESTIONS

Q5 A periodic function $f(x) = x^2$ is defined over an interval of $-2\pi \leq x \leq 2\pi$

(a) Sketch this function over the given interval.

(4 marks)

(b) Prove that the function is an EVEN function.

(4 marks)

(c) Compute the Fourier series for the given function.

(12 marks)

Q6 Consider a wave traveling along a stretched string. The string is attached to rigid surfaces at both ends over a distance of π (mm). The one-dimensional wave equation describes how the displacement $u(x,t)$ of the string or rod varies with position x and time t. The one-dimensional wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 \leq x \leq \pi$$

where c is the wave speed representing how fast the wave propagates. The boundary and initial conditions are given as follows:

<u>Boundary condition:</u>

$u(0, t) = 0 \quad t > 0$

$u(\pi, t) = 0 \quad t > 0$

(a) If the wave speed is taken to be c = 2 m/s, show that the general solution for the equation above can be expressed by:

$$u(x,t) = \sum_{n=1}^{\infty} [b_n \sin(nx)\cos(2nt) + d_n \sin(nx)\sin(2nt)]$$

where b_n and d_n are series constants.

(15 marks)

(b) By taking the following values for b_n and d_n, respectively, obtain the particular solution to the wave equation.

$b_n = \frac{4(-1)^{k-1}}{\pi(2k-1)}$ where $k = 1,2,3 \ldots$

$d_n = 0$

(5 marks)

- END OF QUESTIONS -

APPENDIX A

First Order Differential Equations

Type of ODEs	General solution
Linear ODEs: $y' + P(x)y = Q(x)$	$y = e^{-\int P(x)dx}\left\{\int e^{\int P(x)dx}Q(x)dx + C\right\}$
Exact ODEs: $f(x,y)dx + g(x,y)dy = 0$	$F(x,y) = \int f(x,y)dx$ $F(x,y) - \int \left\{\frac{\partial F}{\partial y} - g(x,y)\right\}dy = C$
Inexact ODEs: $M(x,y)dx + N(x,y)dy = 0$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Integrating factor; $i(x) = e^{\int f(x)dx}$ where $f(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ $i(y) = e^{\int g(y)dy}$ where $g(y) = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$	$\int iM(x,y)dx - \int \left\{\frac{\partial\left(\int iM(x,y)dx\right)}{\partial y} - iN(x,y)\right\}dy = C$

Characteristic Equation and General Solution for Second Order Differential Equations

Types of Roots	General Solution
Real and Distinct Roots: m_1 and m_2	$y = c_1e^{m_1x} + c_2e^{m_2x}$
Real and Repeated Roots: $m_1 = m_2 = m$	$y = c_1e^{mx} + c_2xe^{mx}$
Complex Conjugate Roots: $m = \alpha \pm i\beta$	$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

Method of Undetermined Coefficients

$g(x)$	y_p
Polynomial: $P_n(x) = a_nx^n + \ldots + a_1x + a_0$	$x^r(A_nx^n + \ldots + A_1x + A_0)$
Exponential: e^{ax}	$x^r(Ae^{ax})$
Sine or Cosine: $\cos \beta x$ or $\sin \beta x$	$x^r(A\cos \beta x + B\sin \beta x)$

***Note*:** r is 0, 1, 2 … in such a way that no terms in $y_p(x)$ is similar to those in $y_c(x)$.

Method of Variation of Parameters

The particular solution for $y'' + by' + cy = g(x)$ with b and c as constants is:
$y_p(x) = u_1 y_1 + u_2 y_2$

where,
$u_1 = -\int \frac{y_2 g(x)}{W} dx$
and
$u_2 = \int \frac{y_1 g(x)}{W} dx$

where,
$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$

Laplace Transform

$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = F(s)$	
$f(t)$	$F(s)$
a	$\frac{a}{s}$
$t^n, n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$e^{at}f(t)$	$F(s-a)$
$t^n f(t), n = 1, 2, 3, \ldots$	$(-1)^n \frac{d^n F(s)}{ds^n}$

Laplace Transform (continued)

$H(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)H(t-a)$	$e^{-as}F(s)$
$f(t)\delta(t-a)$	$e^{-as}f(a)$
$y(t)$	$Y(s)$
$\dot{y}(t)$	$sY(s)-y(0)$
$\ddot{y}(t)$	$s^2Y(s)-sy(0)-y'(0)$

Partial Fraction Expansion

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k},\ k=1,2,3,\ldots$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k},\ k=1,2,3,\ldots$

Fourier Series

<u>Fourier series expansion of periodic function with period 2π</u>

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

<u>Half Range Series</u>

$$a_0=\frac{2}{L}\int_0^L f(x)dx$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$$

Trigonometric Identities

TANGENT IDENTITIES

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\cot\theta = \frac{\cos\theta}{\sin\theta}$

RECIPROCAL IDENTITIES

$\csc\theta = \frac{1}{\sin\theta}$ $\sin\theta = \frac{1}{\csc\theta}$

$\sec\theta = \frac{1}{\cos\theta}$ $\cos\theta = \frac{1}{\sec\theta}$

$\cot\theta = \frac{1}{\tan\theta}$ $\tan\theta = \frac{1}{\cot\theta}$

PYTHAGOREAN IDENTITIES

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

PERIODIC IDENTITIES

$\sin(\theta + 2\pi n) = \sin\theta$

$\cos(\theta + 2\pi n) = \cos\theta$

$\tan(\theta + \pi n) = \tan\theta$

$\csc(\theta + 2\pi n) = \csc\theta$

$\sec(\theta + 2\pi n) = \sec\theta$

$\cot(\theta + \pi n) = \cot\theta$

EVEN/ODD IDENTITIES

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

$\csc(-\theta) = -\csc\theta$

$\sec(-\theta) = \sec\theta$

$\cot(-\theta) = -\cot\theta$

DOUBLE ANGLE IDENTITIES

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$= 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta$

$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

HALF ANGLE IDENTITIES

$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$

$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$

$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

LAW OF COSINES

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

$b^2 = a^2 + c^2 - 2ac\cos\beta$

$c^2 = a^2 + b^2 - 2ab\cos\gamma$

PRODUCT TO SUM IDENTITIES

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

SUM TO PRODUCT IDENTITIES

$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

LAW OF TANGENTS

$\frac{a-b}{a+b} = \frac{\tan\left[\frac{1}{2}(\alpha-\beta)\right]}{\tan\left[\frac{1}{2}(\alpha+\beta)\right]}$

$\frac{b-c}{b+c} = \frac{\tan\left[\frac{1}{2}(\beta-\gamma)\right]}{\tan\left[\frac{1}{2}(\beta+\gamma)\right]}$

$\frac{a-c}{a+c} = \frac{\tan\left[\frac{1}{2}(\alpha-\gamma)\right]}{\tan\left[\frac{1}{2}(\alpha+\gamma)\right]}$

SUM/DIFFERENCES IDENTITIES

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

MOLLWEIDE'S FORMULA

$\frac{a+b}{c} = \frac{\cos\left[\frac{1}{2}(\alpha-\beta)\right]}{\sin\left(\frac{1}{2}\gamma\right)}$

LAW OF SINES

$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$

COFUNCTION IDENTITIES

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$

$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$

$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$