UNIVERSITI TUN HUSSEIN ONN MALAYSIA

FINAL EXAMINATION
SEMESTER II
SESSION 2023/2024

COURSE NAME	:	MULTIVARIABLE CALCULUS/ ENGINEERING MATHEMATICS III
COURSE CODE	:	BEE 20303/BEE 21503
PROGRAMME CODE	:	BEJ/BEV
EXAMINATION DATE	:	JULY 2024
DURATION	:	3 HOURS
INSTRUCTIONS	:	1. ANSWER ALL QUESTIONS 2. THIS FINAL EXAMINATION IS CONDUCTED VIA ☐ Open book ☒ Closed book 3. STUDENTS ARE **PROHIBITED** TO CONSULT THEIR OWN MATERIAL OR ANY EXTERNAL RESOURCES DURING THE EXAMINATION CONDUCTED VIA CLOSED BOOK

THIS QUESTION PAPER CONSISTS OF **EIGHT (8)** PAGES

Q1 (a) Use implicit partial differentiation to determine expressions for $\frac{\partial y}{\partial x}$ in the following cases:

(i) $x^5 + 2y^2 = 3xy^4$

(3 marks)

(ii) $\sin^2 x - 6\sin x \cos y = \tan y$

(5 marks)

(b) For a differentiable function $f(x,y)$ with $x = r\cos\theta$ and $y = r\sin\theta$;

(i) Draw the Tree diagram; and

(2 marks)

(ii) Write down the Chain Rule from the Tree diagram and show that: $f_r = f_x \cos\theta + f_y \sin\theta$.

(5 marks)

(c) Calculate the volume bounded by the surfaces $2x + y + z = 4$, planes $x = 0, y = 0$ and $z = 0$. Sketch the 3-dimensional graph and projection on xy-plane to support your answer.

(10 marks)

Q2 (a) Provide **TWO (2)** examples of applications that use triple integration in calculus.

(2 marks)

(b) Evaluate the following triple integral:

$$\int_0^2 \int_0^{3x} \int_0^{y-5} (6z + 1)\, dz\, dy\, dx$$

(5 marks)

(c) A solid tetrahedron bounded by the following planes,

$$x + 2y + z = 4$$

$$x = y$$

$$x = 0\,, z = 0$$

(i) Sketch the surface and the planes in a 3-dimensional Cartesian coordinate system.

(1 mark)

(ii) Sketch the 2-dimensional projection on xy-plane.

(1 mark)

(iii) Compute the volume of the solid bounded by the surface and the planes.

(6 marks)

(d) The solid region Q is the tetrahedron bounded by $x + 2y + 3z = 6$ and has a density, $\rho = 1$. Show that the total mass, m of the solid region Q is 6.

(10 marks)

Q3 (a) Differentiate between scalar field and vector and give an example each.

(5 marks)

(b) Define the gradient of a scalar field in your own words.

(3 marks)

(c) Find the gradient of $f(x, y)$ below and state if the output is a scalar field or vector field.

$$f(x, y) = 2x + xy$$

(5 marks)

(d) Discuss the divergence of a vector field in your own words and support your answer with an example.

(5 marks)

(e) Based on the understanding from **Q3(d)**:

(i) Calculate the divergence of the vector fields $\boldsymbol{f}$ and $\boldsymbol{g}$ below:

$$\boldsymbol{f}(x, y) = x\,\mathbf{i} + y\,\mathbf{j}$$

$$\boldsymbol{g}(x, y) = y\,\mathbf{i} - x\,\mathbf{j}$$

(5 marks)

(ii) From the answers in **Q3(e)(i)**, deduce whether the fields are increasing in strength, decreasing in strength or not changing in overall strength.

(2 marks)

Q4 Given that σ are the surfaces of the cube bounded by $-2 \leq x \leq 2$, $-2 \leq y \leq 2$ and $-2 \leq z \leq 2$, oriented outward and $\boldsymbol{f}(x, y, z) = 5z\,\hat{z}$ is the vector field across σ.

(a) Sketch the cube and show the unit vector normal, $\hat{n}$ from each surface.

(5 marks)

(b) Calculate the flux integral (or also known as surface integral) for surfaces σ_1, σ_3, and σ_5 where:

(i) σ_1 is at positive xy-plane.

(3 marks)

(ii) σ_3 is at positive xz-plane.

(3 marks)

(iii) σ_5 is at positive yz-plane.

(3 marks)

(c) Calculate total flux integral for the surface σ using the Divergence Theorem.

(5 marks)

(d) List the benefits of using the divergence theorem over normal surface integral.

(3 marks)

(e) State the condition of the surface in order to use the Divergence Theorem.

(3 marks)

- END OF QUESTIONS -

APPENDIX A

FORMULAS

Polar coordinate

$x = r\cos\theta,\ y = r\sin\theta,\ \theta = \tan^{-1}(y/x)$, and $\iint_R f(x,y)dA = \iint_R f(r,\theta)\, r\, dr\, d\theta$

Cylindrical coordinate

$x = r\cos\theta,\ y = r\sin\theta,\ z = z$ and $\iiint_G f(x,y,z)\,dV = \iiint_G f(r,\theta,z)\, r\, dz\, dr\, d\theta$

Spherical coordinate

$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$, then $x^2 + y^2 + z^2 = \rho^2$, for $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, and $\iiint_G f(x,y,z)dV = \iiint_G f(\rho,\phi,\theta)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

$$A = \iint_R dA$$

$m = \iint_R \delta(x,y)dA$, where $\delta(x,y)$ is a density of lamina

$$V = \iint_R f(x,y)\,dA$$

$$V = \iiint_G dV$$

$$m = \iiint_G \delta(x,y,z)dV$$

If f is a differentiable function of x, y and z, then the

Grad $f(x,\ y,\ z) = \nabla f(x,\ y,\ z) = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k}$

If $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field in Cartesian coordinates, then the:

Div $\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} + \dfrac{\partial P}{\partial z}$

Curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left(\dfrac{\partial P}{\partial y} - \dfrac{\partial N}{\partial Z}\right)\mathbf{i} - \left(\dfrac{\partial P}{\partial x} - \dfrac{\partial M}{\partial Z}\right)\mathbf{j} + \left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)\mathbf{k}$

F is a conservative vector field if Curl of $\mathbf{F} = 0$.

Surface Integral

Let S be a surface with equation $z = g(x, y)$ and let R be its projection on the xy-plane.

$$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

Gauss's Theorem

$$\iint_S \mathbf{F}\bullet\mathbf{n}\,dS = \iiint_G \nabla\bullet\mathbf{F}\,dV$$

Stokes' Theorem

$$\iint_S (\nabla\times\mathbf{F})\bullet\mathbf{n}\,dS = \oint_C \mathbf{F}\bullet dr$$

Unit normal vector

Outward

$$\mathbf{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{-\frac{\partial z}{\partial x}\mathbf{i}-\frac{\partial z}{\partial y}\mathbf{j}+\mathbf{k}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}}$$

Downward

$$\mathbf{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{\frac{\partial z}{\partial x}\mathbf{i}+\frac{\partial z}{\partial y}\mathbf{j}-\mathbf{k}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}}$$

Identities of Trigonometry and Hyperbolic

Trigonometric Functions

$$\cos^2 x + \sin^2 x = 1$$
$$\sin 2x = 2\sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$2\sin ax \cos bx = \sin(a+b)x + \sin(a-b)x$$
$$2\sin ax \sin bx = \cos(a-b)x - \cos(a+b)x$$

Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh 2x = 2\sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$= 2\cosh^2 x - 1$$
$$= 1 + 2\sinh^2 x$$
$$1 - \tanh^2 x = \text{sech}^2 x$$
$$\coth^2 x - 1 = \text{csch}^2 x$$
$$\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$$
$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$
$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

Implicit Partial Differentiation

$$\frac{\partial z}{\partial x} = -\frac{f_x(x,y,z)}{f_z(x,y,z)}, \qquad \frac{\partial z}{\partial y} = -\frac{f_y(x,y,z)}{f_z(x,y,z)}$$

The derivative of f(x) with respect to x

$$f_x(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Indefinite Integrals and Integration of Inverse Functions

Indefinite Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int e^x dx = e^x + C$$

$$\int \cosh x dx = \sinh x + C$$

$$\int \sinh x dx = \cosh x + C$$

$$\int \text{sech}^2 x dx = \tanh x + C$$

$$\int \text{csch}^2 x dx = -\coth x + C$$

$$\int \text{sech}x \tanh x dx = -\text{sech}x + C$$

Integration of Inverse Functions

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \qquad x^2 < a^2$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + C, \qquad x^2 < a^2$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{-1}{a^2 + x^2} dx = \cot^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{|x|\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + C, \qquad x^2 > a^2$$

$$\int \frac{-1}{|x|\sqrt{x^2 - a^2}} dx = \frac{1}{a} \csc^{-1}\left(\frac{x}{a}\right) + C, \qquad x^2 > a^2$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + C, \qquad x > a > 0$$

$$\int \frac{-1}{|x|\sqrt{a^2 - x^2}} dx = \frac{1}{a} \text{sech}^{-1}\left|\frac{x}{a}\right| + C, \qquad 0 < x < a$$

$$\int \frac{-1}{|x|\sqrt{a^2 + x^2}} dx = \frac{1}{a} \text{csch}^{-1}\left|\frac{x}{a}\right| + C, \qquad x \neq 0$$

$$\int \frac{1}{a^2 - x^2} dx = \begin{cases} \frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right) + C, & x^2 < a^2 \\ \frac{1}{a} \coth^{-1}\left(\frac{x}{a}\right) + C, & x^2 > a^2 \end{cases}$$