Universiti Tun Hussein Onn Malaysia

UNIVERSITI TUN HUSSEIN ONN MALAYSIA

FINAL EXAMINATION
SEMESTER II
SESSION 2023/2024

COURSE NAME	:	ORDINARY DIFFERENTIAL EQUATIONS
COURSE CODE	:	DAU 34403
PROGRAMME CODE	:	DAU
EXAMINATION DATE	:	JULY 2024
DURATION	:	3 HOURS
INSTRUCTIONS	:	1. ANSWER ALL QUESTIONS 2. THIS FINAL EXAMINATION IS CONDUCTED VIA ☐ Open book ☒ Closed book 3. STUDENTS ARE **PROHIBITED** TO CONSULT THEIR OWN MATERIAL OR ANY EXTERNAL RESOURCES DURING THE EXAMINATION CONDUCTED VIA CLOSED BOOK

THIS QUESTION PAPER CONSISTS OF **SIX (6)** PAGES

Q1 (a) A bowl of porridge is placed in a room with temperature of 25°C. The porridge has cooled from 80°C to 40°C after 15 minutes. The process satisfies the Newton's law of cooling that is given by

$$\frac{dT}{dt} = -k(T - T_s),$$

where T is the temperature of the porridge at time t (minute), T_s is the room temperature and k is a constant.

(i) Use separable method to show that the solution of $T(t)$ is given by

$$T(t) = T_s + Ae^{-kt},$$

where $A = e^C$ is a constant.

(5 marks)

(ii) From **Q1(a)(i)**, determine the temperature of the porridge after 60 minutes.

(8 marks)

(b) The number of bacteria in a bottle of yogurt kept in a fridge is represented by the following differential equation

$$\frac{dP}{dt} = 0.37P,$$

where $P(t)$ is the number of bacteria at time t (hour).

(i) By integrating the separable equation, find the general solution of $P(t)$.

(5 marks)

(ii) If the initial number of bacteria at 2.00 am ($t = 0$) is 200, at what time the number of bacteria will reach 1000?

(7 marks)

Q2 (a) Find the solution for the following boundary value problem using the method of undetermined coefficients.

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 2x, \qquad y(0) = 0, \qquad \text{and} \qquad y(4) = 0.$$

(11 marks)

(b) Using method of variation of parameters, solve the following non-homogeneous second order differential equation.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}.$$

(14 marks)

Q3 (a) Find the Laplace transform for $f(t) = 5 - 2t^3 + \cos 4t$.

(4 marks)

(b) Use multiply with t^n property to find the Laplace transform for $f(t) = t\cosh 2t$.

(4 marks)

(c) Use first shift theorem to find the Laplace transform for $f(t) = e^{-2t}\sin 3t$.

(3 marks)

(d) Determine the inverse Laplace transform for the following.

(i) $\dfrac{2\pi}{s} - \dfrac{4}{s^2} + \dfrac{5}{s+1}$.

(3 marks)

(ii) $\dfrac{9s}{s^2+4} - \dfrac{4}{s^2-4}$.

(3 marks)

(e) Use the first shift theorem to obtain the inverse Laplace transform for

$$\frac{3}{(s-9)^4}.$$

(3 marks)

(f) Given

$$F(s) = \frac{3s-5}{s^2-4s+3}.$$

(i) Express $F(s)$ in partial fraction.

(3 marks)

(ii) Hence, obtain the inverse Laplace transform for $F(s)$.

(2 marks)

Q4 (a) Use Laplace transform to solve the initial value problem

$$y' + y = te^{-t}, \qquad y(0) = 2.$$

(10 marks)

(b) (i) Show that,

$$\frac{s^2-8}{(s+3)(s^2-3s+2)} = \frac{1}{20(s-3)} + \frac{7}{4(s-1)} - \frac{4}{5(s-2)}.$$

(5 marks)

(ii) Hence, from **Q4(b)(i)**, solve the initial value problem,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{-3t}, \qquad y(0) = 1, \qquad \text{and} \qquad y'(0) = 0.$$

(10 marks)

- END OF QUESTIONS -

FORMULAE

Table 1: Laplace and Inverse Laplace Transform

Definition: $L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s)$	
$f(t)$	$F(s)$
k	$\frac{k}{s}$
$t^n, n = 1,2,\ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
First Shift Theorem	
$e^{at}f(t)$	$F(s-a)$
Multiply with t^n	
$t^n f(t), n = 1,2,\ldots$	$(-1)\frac{d^nF(s)}{ds^n}$
Initial Value Problem	
$y(t)$	$Y(s)$
$y'(t)$	$sY(s) - y(0)$
$y''(t)$	$s^2Y(s) - sy(0) - y'(0)$

Table 2: Differentiation

$\frac{d}{dx}[k]=0,\quad k$ is a constant	$\frac{d}{dx}[\sec x]=\sec x\tan x$
$\frac{d}{dx}[x^n]=nx^{n-1}$	$\frac{d}{dx}[\operatorname{cosec} x]=-\operatorname{cosec} x\cot x$
$\frac{d}{dx}[e^x]=e^x$	$\frac{d}{dx}[\cot x]=-\operatorname{cosec}^2 x$
$\frac{d}{dx}[\ln\lvert x\rvert]=\frac{1}{x}$	$\frac{d}{dx}[f(u(x))]=\frac{df}{du}\frac{du}{dx}$
$\frac{d}{dx}[\cos x]=-\sin x$	$\frac{d}{dx}[u\pm v]=\frac{du}{dx}\pm\frac{dv}{dx}$
$\frac{d}{dx}[\sin x]=\cos x$	$\frac{d}{dx}[uv]=u\frac{dv}{dx}+v\frac{du}{dx}$
$\frac{d}{dx}[\tan x]=\sec^2 x$	$\frac{d}{dx}\left[\frac{u}{v}\right]=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

Table 3: Integration

$\int k\,dx=kx+C,\quad k$ is a constant	$\int \sin ax\,dx=-\frac{1}{a}\cos ax+C$
$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C$	$\int \cos ax\,dx=\frac{1}{a}\sin ax+C$
$\int \frac{1}{x}dx=\ln\lvert x\rvert+C$	$\int e^{ax}\,dx=\frac{1}{a}e^{ax}+C$
$\int \frac{1}{a+bx}dx=\frac{1}{b}\ln\lvert a+bx\rvert+C$	$\int u\,dv=uv-\int v\,du$

Table 4: Characteristic Equation and General Solution

Homogeneous Differential Equation: $ay''+by'+cy=0$

Characteristic Equation: $am^2+bm+c=0$

$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Case	Roots of Characteristic Equation	General Solution
1	Real and Distinct: $m_1\neq m_2$	$y_h(x)=Ae^{m_1x}+Be^{m_2x}$
2	Real and Equal: $m_1=m_2=m$	$y_h(x)=(A+Bx)e^{mx}$
3	Complex Roots: $m=\alpha\pm i\beta$	$y_h(x)=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$

Table 5: Particular Solution of Nonhomogeneous Equation

Nonhomogeneous Differential Equation: $ay'' + by' + cy = f(x)$	
$f(x)$	$y_p(x)$
$P_n(x) = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$	$x^r(B_n x^n + B_{n-1}x^{n-1} + \cdots + B_1 x + B_0)$
Ce^{ax}	$x^r(Pe^{ax})$
$C\cos\beta x$ or $C\sin\beta x$	$x^r(P\cos\beta x + Q\sin\beta x)$
$P_n(x)e^{ax}$	$x^r(B_n x^n + B_{n-1}x^{n-1} + \cdots + B_1 x + B_0)e^{ax}$
$P_n(x)\begin{cases}\cos\beta x \\ \sin\beta x\end{cases}$ or	$x^r(B_n x^n + B_{n-1}x^{n-1} + \cdots + B_1 x + B_0)\cos\beta x$ + $x^r(B_n x^n + B_{n-1}x^{n-1} + \cdots + B_1 x + B_0)\sin\beta x$
Notes: r is the smallest non-negative integer to ensure no alike term between $y_p(x)$ and $y_h(x)$	

Table 6: Variation of Parameters Method

Homogeneous solution: $y_h(x) = Ay_1 + By_2$
Wronskian function, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$
$u_1 = -\int \frac{y_2 f(x)}{aW}dx + A, \quad u_2 = \int \frac{y_1 f(x)}{aW}dx + B,$
General solution, $y(x) = u_1 y_1 + u_2 y_2$

Table 7: Partial Fraction

$\frac{a}{(s+b)(s-c)} = \frac{A}{(s+b)} + \frac{B}{(s-c)}$
$\frac{a}{s(s-b)(s-c)} = \frac{A}{s} + \frac{B}{(s-b)} + \frac{C}{(s-c)}$
$\frac{a}{(s+b)^2} = \frac{A}{(s+b)} + \frac{B}{(s+b)^2}$
$\frac{a}{(s+b)(s^2+c)} = \frac{A}{(s+b)} + \frac{Bs+C}{(s^2+c)}$